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Introduction to String and Superstring Theory II

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1. Introduction

In the first half of this course, Michael Green set out the historical background of string theory and the basic principles of string mechanics. He described the various consistent string theories and displayed the spectrum of particles which they produce. These always include candidates for gravitons and gauge bosons; in the case of supersymmetric strings, one also finds candidates for approximately massless quarks and leptons. Thus, string theories provide a new basis for understanding the fundamental interactions, a generalized mechanics from which the dynamics of gauge-invariant fields can be derived as a low-energy limit.

Finding the correct particle spectrum is, of course, only the first step toward unravelling the full structure of this grand theory. One would like to understand the origin of the symmetries of the theory, and to realize solutions of the theory which display the observed interactions of particle physics. These deeper levels of analysis of the string theory are still not completed; indeed, they constitute one of the most exciting areas of research in mathematical physics. In this set of lectures I will continue the elucidation of string theories, in a direction that will shed some light on these large issues. The main topic of these lectures will be the formulation of manifestly Lorentz-covariant methods of calculation for string scattering amplitudes. We will find, though, that this study leads us directly to consideration of the gauge invariances of string theory, and to some tools which illuminate the construction of schemes of compactification.

The plan of these lectures is the following: In order to progress in understanding string theory, we must first retrace our steps a bit and review some elements of the quantum theory of massless fields in 2 dimensions. The string world sheet is, of course, a 2-dimensional surface, and the displacements of the string in space-time can well be viewed as (massless) fields on this surface. We have seen in Michael Green's lectures that the conformal invariance of 2-dimensionsal massless fields gives rise to important simplifications in the calculations of string amplitudes. In this second half of the course, I would like to elevate conformal invariance to a guiding principle for the construction of string theories. It will then be very useful to formulate 2-dimensional massless field theories in such a

way that their conformal invariance is manifest. This was done in a very beautiful way by Belavin, Polyakov, and Zamolodchikov.[1] In Section 2, I will review the formalism which these authors have presented; this formalism will provide the basic language for our later arguments. Our use of conformal field theory methods in application to string theory follows the work of Friedan, Martinec, and Shenker;[2,3] Our general development, and especially our treatment of the critical dimensions and the formalism for the scattering amplitudes of strings and superstrings, follows their approach closely.

The discussion of string theory proper will begin in Section 3. Here, I will use conformal invariance to rederive the basic results on the embedding dimensionality for bosonic and fermionic strings. Section 4 will discuss the spectrum of the bosonic string and the computation of scattering amplitudes. In Section 5, I will extend this formalism to clarify the origin of Yang-Mills gauge invariance in the open bosonic string theory. Section 6 will address the question of the general-coordinate gauge invariance of string theory, presenting two disparate points of view on this question.

In Section 7, I will analyze the superstring theory from the viewpoint of 2-dimensional conformal invariance. I will rederive the basic results on the particle spectrum and present methods for the covariant calculation of superstring scattering amplitudes. In Section 8, I will discuss the 1-loop amplitudes of bosonic and supersymmetric string theories.

The last two sections will give a brief introduction to some of the deeper questions of the theory, especially the question of the reduction from the idealized string theory in 10 extended dimensions to more realistic solutions in which all but 4 of these dimensions are compactified. In Section 9, I will outline briefly what is known about the space-time supersymmetry of the superstring from the covariant viewpoint. I will then present a precis of the approach advanced by Candelas, Horowitz, Strominger, and Witten[4] for identifying possible 6-dimensional spaces which might represent the form of the compact dimensions. Section 10 will give a somewhat more detailed presentation of the *orbifold* scheme of compactification suggested by Dixon, Harvey, Vafa, and Witten.[5] This scheme has the advantage of allowing explicit calculations of many aspects of the conpactified theory, and we will find it illuminating to carry through a part of this analysis.

This full course of lectures is still far from comprising a complete summary of knowledge on string theory.* This set of lectures will certainly raise as many questions as it answers. I hope that you, the reader, will be intrigued to seek the

* A different, and more extensive, selection of topics is given in the new book of Green, Schwarz, and Witten.[6]

answers to these questions, and thus to join the quest for understanding of this most promising and mysterious branch of fundamental theory.

2. Conformal Field Theory

The main goal of this set of lectures will be to reconstruct the spectrum and interactions for bosonic and supersymmetric strings using as our primary tool the conformal symmetry of the dynamics on the string world sheet. It is therefore appropriate that we begin our discussion by setting out the formalism of 2-dimensional conformal invariance. Conformal symmetry is, exactly as it implies, the symmetry of conformal mappings of the 2-dimensional plane. One should then naturally expect that a conformally invariant theory will involve fields which are analytic functions of the 2-dimensional coordinates treated as a complex variable. Recently, Belavin, Polyakov, and Zamolodchikov,[1] (BPZ), building on results from the early period of string theories,[7-9] have shown how to write conformally invariant theories explicitly in terms of analytic fields. In this section, I will review this beautiful formalism, which will provide a natural language for our subsequent exploration of string theory.

2.1. Conformal Coordinates

To begin our study, we must define the basic coordinates. Throughout these lectures, I will describe both the string world surface and the spacetime in which it is embedded by their Euclidean continuations. On the string world surface, this continuation corresponds to

$$(\tau \pm \sigma) \;\rightarrow\; -i\,(\tau \pm i\sigma)\;. \tag{2.1}$$

Let us then define

$$w \;=\; \tau + i\sigma\;, \qquad w \;=\; \tau - i\sigma\;. \tag{2.2}$$

The decomposition of a string state into running waves moving to the left or to the right around the string becomes, after this continuation, a decomposition into analytic and anti-analytic functions on the 2-dimensional Euclidean surface. The Euclidean string covers only a finite interval of σ and therefore only a strip of the 2-dimensional plane, shown in Fig. 1(a). However, if we anticipate that the theory

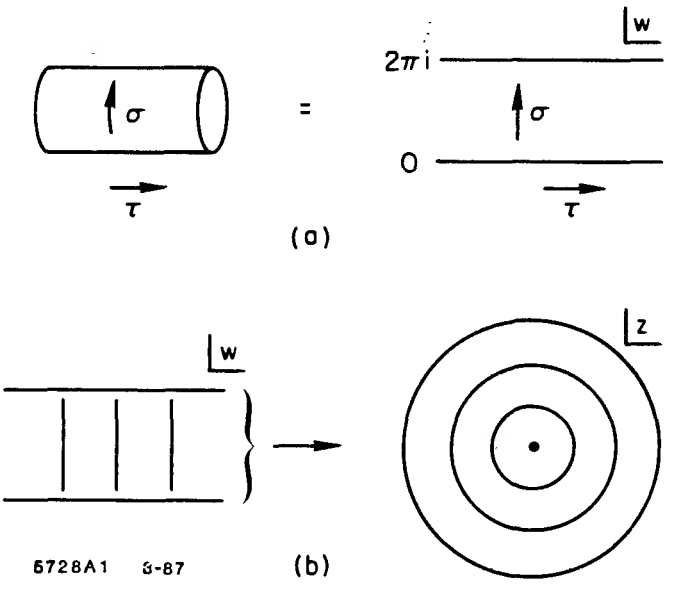

Figure 1. The string world sheet considered as a region of the complex plane (a) in the original variables τ and σ, (b) in the variable $z = \exp(\tau + i\sigma)$.

we will construct will have complete symmetry under conformal transformations, we can map this region into the whole complex plane by the mapping

$$z = \exp(\tau + i\sigma) \ . \tag{2.3}$$

The form of this mapping is shown in Fig. 1(b). Lines of constant τ are mapped into circles on the z plane; the operation of time translation, $\tau \to \tau + a$, becomes the dilatation,

$$z \to e^a z \ . \tag{2.4}$$

The string coordinates $X^\mu(\tau, \sigma)$, separated into left- and right-moving excitations, correspond to analytic and anti-analytic fields in this complex z plane. We must address the question of whether a sensible Euclidean quantum theory of analytic fields can be defined. A part of this definition must include the identification of operators which implement conformal mappings of the z plane. Within the family of such operators, we should identify the generator of dilatations (2.4) with the Hamiltonian of the original string theory. Such a procedure of identifying dilatations with the Hamiltonian and circles about the origin with equal-time surfaces is called *radial quantization.*

Let us begin to piece together this formalism. To begin, assume that the z plane is a flat Euclidean space, so that its metric is $g_{\alpha\beta} = \delta_{\alpha\beta}$. In complex coordinates,

$$g_{z\bar{z}} = \frac{1}{2}\,, \quad g_{zz} \;=\; g_{\bar{z}\bar{z}} \;=\; 0\,. \tag{2.5}$$

Assume that we can at least construct a theory on this plane which is scale invariant. Let us work out the consequences of this statement. Since the energy-momentum tensor $T_{\alpha\beta}$ generates local translations and a dilatation corresponds to a local motion $x^\alpha \to x^\alpha + \delta\lambda \cdot x^\alpha$, the dilatation current should be just

$$D_\alpha \;=\; T_{\alpha\beta}x^\beta\,. \tag{2.6}$$

Since the energy-momemtum tensor is conserved, the statement of scale invariance

$$\partial_\alpha D^\alpha \;=\; 0 \qquad \text{implies} \qquad T_\alpha{}^\alpha \;=\; 0\,. \tag{2.7}$$

Up to this point, these statements are true in any dimension. In 2 dimensions, however, when we use complex coordinates, (2.7) takes the following form:

$$T_{z\bar{z}} \;=\; 0\,. \tag{2.8}$$

But then we can use the equation of energy-momentum conservation

$$\partial_{\bar{z}}T_{zz} \;+\; \partial_z T_{\bar{z}z} \;=\; 0 \tag{2.9}$$

to prove that

$$\partial_{\bar{z}}T_{zz} \;=\; 0\,. \tag{2.10}$$

Thus, the field $T_{zz} \equiv T$ is an analytic function of z. Similarly, $T_{\bar{z}\bar{z}} \equiv \overline{T}$ is depends only on $\bar{z}$ and so is an anti-analytic field.

2.2. Conformal Transformations

It is not unreasonable to expect T and $\overline{T}$, as the remnants of the energy momentum tensor in these complex coordinates, to generate local conformal transformations. Let us try to formulate this conjecture more precisely. A natural

form for the operator which generates the infinitesimal conformal mapping

$$z \rightarrow z + \epsilon^z(z) \tag{2.11}$$

would be $\int dz\, \epsilon^z(z)\, T_{zz}(z)$; the integral should be taken over an equal-time surface, that is, one of the circles shown in Fig. 1(b). Let us, then, define

$$T_\epsilon = \oint \frac{dz}{2\pi i}\, \epsilon(z)\, T(z) , \tag{2.12}$$

where the path of integration is a circle about the origin. Note that, since the integrand is an analytic function, we may move the integral from any such circle to any other by a contour deformation without changing its value; thus, T_ϵ is a conserved charge. We would expect that the transformation (2.11) would be implemented in a quantum theory in the form of the commutators of local fields with (2.12).

We can easily check this conjecture for the simple case of a free massless scalar field. Write the action for such a field as

$$\int \mathcal{L} = \frac{1}{2\pi} \int d^2z\, \partial_z X \partial_{\bar{z}} X . \tag{2.13}$$

The propagator of this field is the Green's function for the Laplace equation in 2 dimensions: $\langle X(z_1)X(z_2)\rangle = -2\log|z_1 - z_2|$. Split this into the pieces corresponding to the analytic and anti-analytic components. The propagator of the analytic field is then

$$\langle X(z_1)X(z_2)\rangle = -\log(z_1 - z_2) . \tag{2.14}$$

The analytic part of the energy-momentum tensor for this field is given by

$$T_{zz} = -\frac{1}{2} : (\partial_z X)^2 : . \tag{2.15}$$

Henceforth, unless a different convention is indicated explicitly, I will always consider products of analytic fields at the same point to be normal-ordered.

Now we are ready to compute the commutator of the operator (2.12) constructed from (2.15) with some local field operator $\mathcal{O}(z)$. $X(z)$ is actually not a good first choice, since this field can have logarithmic branch cuts (as seen in

(2.14)), so I will choose instead to compute $[T_\epsilon, \partial_z X(z)]$. The contour defining T_ϵ may be taken to be the equal-time circle containing z. We may make use of the analytic properties of $T(z)$ and $\partial_z X(z)$ to evaluate this commutator by relating it to expectation values on the plane. Consider, then, the correlation function defined by functional integration:

$$\langle T_\epsilon \, \partial_z X(z) \rangle \;=\; \frac{1}{Z} \int \mathcal{D}X \; e^{-S} \quad T_\epsilon \, \partial_z X(z) \; . \tag{2.16}$$

The functional integral defines the operator product by setting the operators in *time order*. We may define the equal-time commutator of two operators, then, as the difference of two correlation functions of the form (2.16) in which the operator T_ϵ has been displaced slightly forward and backward in time (that is, in radial distance from the origin) with respect to the point z. Thus, we write

$$\begin{aligned} &\langle [T_\epsilon \, , \, \partial_z X(z)] \rangle \\ &\quad = \frac{1}{Z} \int \mathcal{D}X \; e^{-S} \quad (T_\epsilon(r = |z| + \delta) \, \partial_z X(z) \;-\; T_\epsilon(r = |z| - \delta) \, \partial_z X(z)) \; . \end{aligned} \tag{2.17}$$

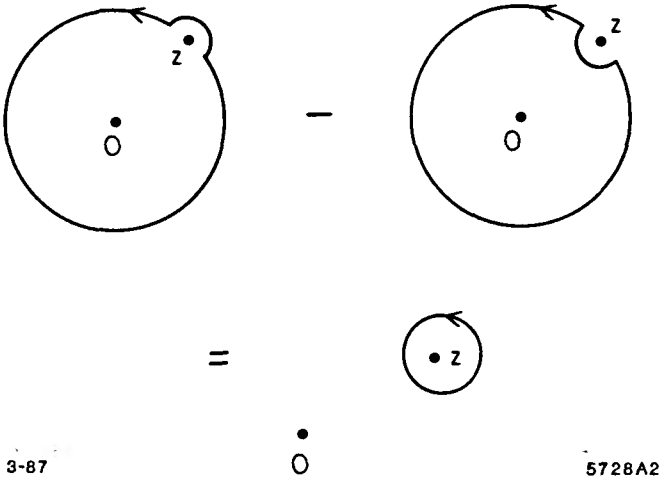

Figure 2. Computation of a commutator in terms of the poles of correlation functions.

The construction is illustrated in Fig. 2. As the figure indicates, T_ϵ is defined as a contour integral; since the integrand is analytic, we may deform the contour except where we encounter a singularity. Thus, by the cancellation shown, the commutator depends only on the singularity of the correlation function of $T(z')$ and $\partial_z X(z)$ in the limit $z' \to z$. This is easy to work out; letting $\langle X(z')X(z)\rangle$ denote the Wick contraction (2.14), we can compute

$$\begin{aligned} T(z')\,\partial_z X(z) &= -\frac{1}{2}\partial_z X\,\partial_z X(z')\quad \partial_z X(z) \\ &= -\frac{1}{2}\partial_z X\,\langle \partial_z X(z')\,\partial_z X(z)\rangle \cdot 2 \;+\; (\text{nonsingular}) \\ &= \quad \partial_z X(z')\,\frac{1}{(z'-z)^2} \;+\; (\text{nonsingular})\,. \end{aligned} \tag{2.18}$$

Taylor-expanding the field about $z' = z$ produces

$$T(z')\,\partial_z X(z) \;\sim\; \frac{1}{(z'-z)^2}\partial_z X(z) \;+\; \frac{1}{(z'-z)}\partial_z^2 X(z) \;+\; \cdots \tag{2.19}$$

This imples

$$\begin{aligned} \langle\, [T_\epsilon,\, \partial_z X(z)]\,\rangle &= \oint \frac{dz'}{2\pi i}\,\epsilon(z')\left[\frac{1}{(z'-z)^2}\partial_z X(z) \;+\; \frac{1}{(z'-z)}\partial_z^2 X(z) \;+\; \cdots\right] \\ &= \partial_z\epsilon\cdot\partial_z X \;+\; \epsilon\cdot\partial_z^2 X\,. \end{aligned} \tag{2.20}$$

Is this a sensible result? Under the transformation $z \to z + \epsilon(z)$, we should expect

$$\begin{aligned} X(z) &\to X(z) \;+\; \epsilon\,\partial_z X(z)\,, \\ \partial_z X(z) &\to \partial_z X(z) \;+\; \partial_z\epsilon\,\partial_z X(z) \;+\epsilon\,\partial_z^2 X(z)\,. \end{aligned} \tag{2.21}$$

We may explain the extra term in the second line by noting that $\partial_z X$ is a tensor, of rank (-1). In general, a tensor transforms under reparametrizations according to

$$t^{\alpha\beta\cdots}(x) \;\to\; \frac{\partial\tilde{x}^\alpha}{\partial x^{\overline{\alpha}}}\cdots t^{\overline{\alpha}\overline{\beta}\cdots}(x(\tilde{x}))\,. \tag{2.22}$$

If we specialize this equation to conformal transformations in 2 dimensions, and

consider tensors with r analytic indices only, this transformation law becomes

$$t^{zz\cdots}(z) \;\to\; \left(\frac{dz}{d\tilde{z}}\right)^r t^{zz\cdots}(\tilde{z}(z)) \;. \tag{2.23}$$

For an infinitesimal transformation, $\tilde{z} = z + \epsilon(z)$,

$$\delta t \;=\; -r\,\partial_z\epsilon\cdot t \;+\; \epsilon\cdot\partial_z t \;; \tag{2.24}$$

this properly reproduces (2.21) and (2.20) for $r = -1$.

Under a dilatation $\tilde{z} = \lambda z$, (2.23) takes the form

$$t \;\to\; \lambda^r t(\lambda^{-1}\tilde{z}) \;, \tag{2.25}$$

so that the rank of an analytic tensor is also its scaling dimension. Normally in particle physics, we quote the dimension in mass units; thus we should write $d = -r$. Assembling the pieces of our analysis, we find that the commutator of the generator of a conformal transformation T_ϵ with a local tensor field of dimension d_t should be

$$[T_\epsilon, t(z)] \;=\; d_t\cdot\partial_z\epsilon\cdot\partial_z t(z) \;+\; \epsilon\cdot\partial_z^2 t(z) \;. \tag{2.26}$$

This commutator follows, by the manipulations described above, from the operator product expansion

$$T(w)\,t(z) \;\sim\; \frac{d_t}{(w-z)^2}\,t(z) \;+\; \frac{1}{w-z}\,\partial_z t(z) \;+\; (\text{nonsingular}) \;. \tag{2.27}$$

This operator product formula encodes the conformal transformation properties of fields in a way which will prove very convenient for mathematical analysis.

To gain a better understanding of the formula (2.27), it will be helpful to work through one more example. Consider the operator $e^{i\alpha\cdot X(z)}$ formed as the exponential of the free field. Let us compute the operator product of the free-field

$T(w)$ with this field:

$$-\frac{1}{2}\big(\partial_w X(w)\big)^2\, e^{i\alpha X(z)} \;\sim\; -\frac{1}{2}\big(\langle \partial_w X(w)\; i\alpha\cdot X(z)\rangle\big)^2\, e^{i\alpha\cdot X(z)}$$
$$-\;\frac{1}{2}\cdot 2\,\partial_w X\,\big(\langle \partial_w X(w)\; i\alpha\cdot X(z)\rangle\big)\, e^{i\alpha\cdot X(z)}$$
$$\sim\; \frac{\alpha^2/2}{(w-z)^2}\, e^{i\alpha\cdot X(z)} \;+\; \frac{i\alpha\cdot\partial_w X}{w-z}\, e^{i\alpha\cdot X(z)} \;+\; \ldots, \tag{2.28}$$

so that

$$-\frac{1}{2}(\partial_w X)^2\, e^{i\alpha\cdot X(z)} \;\sim\; \frac{\alpha^2/2}{(w-z)^2}\, e^{i\alpha\cdot X(z)} \;+\; \frac{1}{w-z}\,\partial_z\Big(e^{i\alpha\cdot X}\Big) \;+\; \ldots \tag{2.29}$$

Thus, $e^{i\alpha\cdot X(z)}$ is a conformal tensor of dimension $\alpha^2/2$. This same scaling dimension can be read from the correlation function

$$\Big\langle e^{i\alpha\cdot X(w)}\; e^{-i\alpha X(z)}\Big\rangle \;=\; \exp[-\alpha^2\log(w-z)] \;=\; (w-z)^{-\alpha^2}\;. \tag{2.30}$$

Not all fields are tensors. Derivatives of tensors, for example, have more complicated transformation laws. BPZ refer to fields with the above transformation laws as *primary fields*; their successive derivatives are called *secondary fields*. In general, the operator product of secondary fields with $T(z)$ has higher singularities than the double pole seen in (2.27).

In addition to analytic fields $t(z)$, transformed by the action of T_{zz}, a conformal field theory will have anti-analytic fields $\bar{t}(\bar{z})$, transformed by the action of $T_{\bar{z}\bar{z}}$. The simplest of these fields will be tensors with $\bar{r}$ anti-analytic indices; these are the primary anti-analytic fields of dimension $\bar{d}_t = -r$. The theory of these tensors and their transformations can be developed precisely in parallel to the discussion of analytic tensors which we have just completed. More generally, we should expect that some primary fields will transform with both analytic and anti-analytic indices; these will be characterized as tensors of dimension $(d_t, \bar{d}_t)$.

2.3. The Conformal Algebra

The elementary infinitesimal conformal transformations corresponding to $\epsilon(z) = z^{n+1}$ are generated by the Fourier components of $T(z)$ on the circle:

$$L_n = \oint \frac{dw}{2\pi i} w^{n+1} T(w) . \tag{2.31}$$

From Michael Green's lectures, you might expect that the L_n are the Virasoro operators.[10] To verify this, let us compute their algebra:

$$[L_n, L_m] = \left[\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right] z^{n+1} T(z)\, w^{m+1} T(w) . \tag{2.32}$$

The change in the order of the z and w integrations represents a small displacement of the z contour outside and then inside the w contour, implementing the functional definition of the commutator that I have described earlier.

The difference of integrals is nonzero only by virtue of the singularity of the operator product as $z \to w$. In the free boson field theory, we can readily compute this singularity:

$$\begin{aligned} T(z)T(w) &= (-\tfrac{1}{2})^2 \cdot 2 \cdot (\langle \partial_z X \partial_w X \rangle)^2 + \\ &\quad + (-\tfrac{1}{2})^2 \cdot 4 \cdot \partial_z X \langle \partial_z X \partial_w X \rangle \partial_w X + \text{(nonsingular)} \\ &= \frac{1}{2} \left(\frac{-1}{(z-w)^2} \right)^2 + \partial_z X \frac{-1}{(z-w)^2} \partial_w X + \dots \\ &= \frac{1}{2} \frac{1}{(z-w)^4} \\ &\quad + \frac{2}{(z-w)^2} \left[-\frac{1}{2} (\partial_w X)^2 \right] + \frac{1}{(z-w)} \partial_w \left[-\frac{1}{2} (\partial_w X)^2 \right] + \dots \end{aligned} \tag{2.33}$$

The last two terms are precisely what would be required for T to transform itself as a tensor of dimension 2 under conformal transformations. (Note that the last term fixes the normalization of $T(z)$.) The first term is an extra c-number—generated as a purely quantum mechanical effect. The form of this term is determined by scale transformation properties: Since T has mass dimension 2,

this term must be a pure power $(z-w)^{-4}$. However, the overall coefficient of this term is not fixed and can vary from system to system. In a general conformal field theory, then, we expect the $T-T$ operator product to take the form

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{(z-w)}\,\partial_w T + \cdots \qquad (2.34)$$

where c is a fixed number.

We can now find the general form of the commutator of two L_n operators by inserting (2.34) into (2.32), and drawing the z contour tightly about the point w. This gives

$$\begin{aligned}
[L_n, L_m] &= \oint \frac{dw}{2\pi i}\, w^{n+1} \oint \frac{dz}{2\pi i}\, z^{n+1} \\
&\qquad \times \left[\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{(z-w)}\,\partial_w T\right] \\
&= \oint \frac{dw}{2\pi i}\, w^{m+1}\Big\{(n+1)w^n \cdot 2T(w) + w^{n+1}\partial_w T \\
&\qquad + (c/2)\cdot\frac{w^{n-2}}{3!}\,(n+1)n(n-1)\Big\} \\
&= \oint \frac{dw}{2\pi i}\,\Big\{(2n+2)w^{n+m+1}T(w) - (m+n+2)w^{n+m+1}T(w) \\
&\qquad + \frac{c}{12}\,n(n+1)(n-1)w^{m+n-1}\Big\}.
\end{aligned} \qquad (2.35)$$

This is just

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{c}{12}\,n(n+1)(n-1)\,\delta(m+n)\,, \qquad (2.36)$$

which is indeed the Virasoro algebra. The constant c is called the *central charge* of the the Virasoro algebra; this takes the value $c = 1$ for one massless scalar field.

We can see from (2.34) that the constant c appears in the vacuum expectation value of the square of the operator T_{zz}. It can be argued on this basis that c must be positive if the underlying Hilbert space has a positive metric.[11] We may derive this result more explicitly as follows: In the next few sections, we will construct a state $|0\rangle$, such that $L_n|0\rangle = 0$ for $n \geq 0$, and an inner product such that $L_n^\dagger = L_{-n}$. Using these ingredients, we may write:

$$\frac{1}{2}c \;=\; \langle 0|\,[L_2,\,L_{-2}]\,|0\rangle \;=\; \langle 0|\,L_2\,L_2^\dagger\,|0\rangle \;>\; 0\;. \tag{2.37}$$

The most important of the L_n is L_0, the generator of dilatations. It should be noted, though, that the central charge term in (2.36) actually vanishes for the set of three generators L_{-1}, L_0, L_1. These operators generate the infinitesimal transformations

$$\delta z \;=\; \alpha \;+\; \beta z \;+\; \gamma z^2 \tag{2.38}$$

From (2.36), we see that these generators form a closed subalgebra. The subgroup of conformal transformations generated by this algebra is the group of fractional linear transformations

$$z \;\rightarrow\; z' \;=\; \left(\frac{az+b}{cz+d}\right)\;. \tag{2.39}$$

(Expanding (2.39) about $a = d = 1$, $b = c = 0$ reproduces (2.38).)

The complementary algebra of anti-analytic transformations is generated by a second set of Virasoro operators

$$\overline{L}_n \;=\; \oint \frac{d\overline{w}}{-2\pi i}\;\overline{w}^{n+1}\overline{T}(\overline{w})\;. \tag{2.40}$$

Just analogously, these generators possess a closed subalgebra generated by $(\overline{L}_{-1}, \overline{L}_0, \overline{L}_1)$ with vanishing central charge. Together, the two sets of operators generate infinitesimal transformations of the form

$$\delta z \;=\; \alpha \;+\; \beta z \;+\; \gamma z^2\;, \qquad \delta\overline{z} \;=\; \overline{\alpha} \;+\; \overline{\beta}\overline{z} \;+\; \overline{\gamma}\overline{z}^2\;, \tag{2.41}$$

acting on all of the operators of the conformal field theory. This transformation is the infinitesimal form of a fractional linear transformation (2.39) with general complex coefficients.

When one composes two mappings of the form (2.39), the parameters of the product mapping are obtained by matrix multiplication of the original parameters. In particular, the determinants $(ad - bc)$ simply multiply. Since one of the four parameters of (2.39) is redundant, we may fix $(ad - bc) = 1$. Then the set of mappings (2.39) with complex coefficients, considered as a group under composition, is isomorphic to the group of 2×2 matrices with unit determinant, $SL(2, C)$. This subgroup of the full conformal group plays an important role in string theory, as we will see in Sections 4 and 5.

The definition of the L_n may be considered as a Fourier analysis of $T(z)$. It will be convenient to introduce a set of conventions for Fourier-analyzing a more general tensor field $t(z)$, of dimension d_t. Let us define

$$t_n = \oint \frac{dz}{2\pi i} z^{n+d_t-1} t(z) , \qquad t(z) = \sum_{n=-\infty}^{\infty} t_n z^{-n-d_t} . \tag{2.42}$$

To see the utility of this definition, compute the commutator of the Fourier component t_n, defined in this way, with L_0. We find

$$\begin{aligned}
[L_0, t_n] &= \left[\oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} - \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \right] wT(w)\, z^{n+d_t-1} t(z) \\
&= \oint \frac{dz}{2\pi i} z^{n+d_t-1} \int \frac{dw}{2\pi i} w \left[\frac{d_t}{(w-z)^2} t(z) + \frac{1}{(w-z)} \partial_z t \right] \\
&= \oint \frac{dz}{2\pi i} z^{n+d_t-1} \left[d_t\, t(z) + z \partial_z t(z) \right] \\
&= -n \oint \frac{dz}{2\pi i} z^{n+d_t-1} t(z) .
\end{aligned} \tag{2.43}$$

Thus,

$$[L_0, t_n] = -n t_n ; \tag{2.44}$$

that is, t_n lowers L_0 by n units. In string theory, we will interpret the dilatation generator L_0 as the Hamiltonian of the single-string dynamics. Fourier components t_n will be annihilation operators for $n > 0$ and creation operators for $n < 0$. Ladder operators of anti-analytic tensors may be defined in an analogous way.

3. Critical Dimensions

In order to connect the conformal field theory on the complex plane defined in the previous section to the dynamics of coordinates $X^\mu(\tau,\sigma)$ on a string, one further condition must be satisfied. The program displayed in Fig. 1 requires that we replace functional integrals over fields on the string surface with functional integrals over fields on the plane. If the field theory of two-dimensional fields on the string is conformally invariant, it would seem that we could freely make this replacement. However, there is a subtlety which we must recognize and deal with. The usual criterion for conformal invariance is that expectation values of (scalar) operators are unchanged by conformal transformations. This leaves over the possibility that the functional integral over fields could change its normalization by a c-number factor when we make a conformal transformation. This factor would disappear when one computes a correlation function. However, in string theory, the string scattering amplitudes will be identified with the functional integrals themselves, and these possible c-number factors will appear explcitly as violations of conformal symmetry. We can take the transformation shown in Fig. 1 absolutely literally, then, only if we can identify and cancel this extra c-number term. In this section, I will discuss that cancellation, which, as we will see, implies that the string must live in a specific critical embedding dimension.

3.1. Conformal Transformations and Conformal Ghosts

We should begin by reviewing the route from a geometrically invariant formulation of the string dynamics to expressions of the form of eq. (2.13) in a fixed background metric. A geometrically invariant expression of the dynamics of the string coordinate field is given by writing[12]

$$Z = \int DXDg\, e^{-\int \mathcal{L}}, \tag{3.1}$$

where

$$\int \mathcal{L} = \frac{1}{8\pi}\int d^2\xi \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\mu \tag{3.2}$$

and the functional integral is taken over both the fields $X^\mu(\xi)$ and the metric $g_{\alpha\beta}(\xi)$ on the world surface. Since the action (3.2) is reparametrization invariant, we are free to change coordinates to simplify the form of the metric. On the plane, or on a region such as the strip of Fig. 1 with the topology of a plane, we may use 2-dimensional reparametrizations to remove two degrees of freedom from $g_{\alpha\beta}$

and convert it to the form

$$g_{\alpha\beta} = e^{\phi(\xi)} \cdot \delta_{\alpha\beta} . \tag{3.3}$$

The special case of (3.3) with $\phi(\xi) = 0$ is exactly the flat metric, eq. (2.5), used in the previous section.

If we insert (3.3) into the action (3.2), $\phi(\xi)$ cancels out. This is exactly the statement that (3.2) is classically conformally invariant. If the metric has the form (3.3) in one coordinate system, a conformal transformation $z \to \tilde{z}(z)$ carries

$$g_{z\bar{z}} \to \left|\frac{d\tilde{z}}{dz}\right|^2 g_{z\bar{z}} , \tag{3.4}$$

so this tranformation causes simply a shift of $\phi(\xi)$. If $\phi(\xi)$ is irrelevant, then, we can move freely between different conformally related coordinate systems. This is what we need to view the strip and the plane shown in Fig. 1 as being completely equivalent.

If indeed the integrand of (3.1) is independent of $\phi(\xi)$, we can treat this variable as a gauge parameter and fix it as the same time that we fix the reparametrization freedom. In this case, the functional integral of eq. (3.1) should more properly be written as

$$Z = \frac{1}{V_{rep} V_{conf}} \int DX Dg \, e^{-\int \mathcal{L}} , \tag{3.5}$$

where the two prefactors represent the volumes of the reparametrization and conformal groups. This expression may be evaluated by using the Fadde'ev-Popov procedure. If we gauge-fix to the coordinate system (2.5), shifting the conformal factor $\phi(\xi)$ simply shifts the diagonal component of the metric; this leads to a trivial Fadde'ev-Popov determinant. The off-diagonal terms in the metric are induced by reparametrizations

$$\delta g_{zz} = \partial_z(\delta\xi^{\bar{z}}) ; \tag{3.6}$$

setting these components equal to zero as a gauge condition leads to a nontrivial determinant

$$\Delta = \det(\partial_z) \cdot \det(\partial_{\bar{z}}) , \tag{3.7}$$

which may be represented by a functional integral over ghost fields. In all, we

find that the gauge-fixing replaces

$$\frac{1}{V_{rep}V_{conf}}\int \mathcal{D}g \;=\; \int \mathcal{D}b\,\mathcal{D}c\; e^{-\int (b_{zz}\partial_{\overline{z}}c^{z}+\overline{b}_{\overline{z}\overline{z}}\partial_{z}\overline{c}^{\overline{z}})}\;, \tag{3.8}$$

where b, c, $\overline{b}$, $\overline{c}$ are anticommuting fields. I have assigned the ghosts c, $\overline{c}$ the transformation properties of reparametrizations $\delta\xi^{z}$, $\delta\xi^{\overline{z}}$; the antighosts have the tranformation laws needed to make the action of (3.8) geometrically invariant.

Unfortunately, the formal property that the integrand of (3.1) does not depend on $\phi(\xi)$ is true only classically and (generally) disappears in the quantum theory. Polyakov[12] discovered that the regulation of the functional integral over $X^{\mu}(\xi)$ breaks the conformal symmetry and leads to a multiplicative c-number dependence on ϕ: Under a shift from $\phi = 0$ to a nonzero $\phi(\xi)$, $Z_X = \int \mathcal{D}X \exp(-\int \mathcal{L})$ tranforms according to

$$Z_X \;\rightarrow\; Z_X \cdot \exp\left[-\frac{D}{96\pi}\int d^2\xi(\partial_\alpha\phi(\xi))^2\right]\,. \tag{3.9}$$

where D is the number of coordinate fields, or, equivalently, the dimension of the space-time in which the string is embedded. This violation of conformal invariance can be understood as a consequence of some considerations of the previous section. Since a nonzero $\phi(\xi)$ can be generated by a conformal transformation, let us look more closely at the algebra of conformal transformations derived there. In particular, let us apply $\partial_{\overline{z}}$ to eq. (2.34). Using

$$2\partial_{\overline{z}}\partial_z \log|z-w| \;=\; \partial_{\overline{z}}\frac{1}{(z-w)} \;=\; \pi\delta^{(2)}(z-w)\;, \tag{3.10}$$

we find

$$\partial_{\overline{z}}T_{zz}(z)\;T_{ww}(w) \;=\; -c\cdot\frac{\pi}{12}\partial_z^3\,\delta^{(2)}(z-w)\;+\cdots \tag{3.11}$$

Thus, the existence of the central charge implies that T_{zz} is not completely analytic. By the logic of eq. (2.10), this implies that the system is not completely scale-invariant. Let us manipulate this relation further by applying $\partial_{\overline{w}}$ and using (2.9) to exchange T_{zz} for $T_{z\overline{z}}$. Then

$$T_{z\overline{z}}(z)\;T_{w\overline{w}}(w) \;=\; c\cdot\frac{\pi}{12}\cdot\partial_z\partial_{\overline{w}}\delta^{(2)}(z-w)\;+\;\cdots \tag{3.12}$$

An infinitesimal shift of ϕ is brings down from the exponent of the functional

integral the term

$$\frac{1}{2\pi}\int d^2\xi\ \phi(\xi)T_{z\bar{z}}(\xi) \tag{3.13}$$

Thus, eq. (3.12) implies that a shift of ϕ produces a multiplicative c-number factor of just the form of (3.9), with coefficient proportional to the central charge c of the Virasoro algebra.[13]

In many contexts, the conformal transformation law (3.9) is perfectly acceptable. In a few cases, it is actually physically relevant; for example, in a statistical mechanical system on a strip of finite width, mapping the strip to the full plane and applying (3.9) gives the correct dependence of the free energy on the finite size of the strip.[14,15] We have already remarked, however, that the standard formulation of string theory requires that the conformal motion be a gauge symmetry of (3.1). The extra term in (3.9) is unacceptable. This higher criterion of conformal invariance imposes an additional stringent constraint on a conformal field theory: The central charge c of the Virasoro algebra must vanish identically. This criterion is made especially difficult to fulfill by two observations made in the previous section: First, the coordinate fields X^μ each contribute one unit to c, for a total of $c = D$. Second, an additional field on the world sheet can give a negative, cancelling, contribution to c only if it creates states of negative metric.

Polyakov realized that this apparent dilemma has a very natural resolution. The reparametrization ghosts introduced in (3.8) create negative-metric states and must, in any case, be included in any analysis of the conformal transformation properties of the string functional integral. Let us, then, set up the conformal field theory for these ghost fields and compute their contribution to c.

The fields c^z and b_{zz} obey the classical equation of motion

$$\partial_{\bar{z}}c^z \ = \ \partial_{\bar{z}}b_{zz} \ = \ 0\ . \tag{3.14}$$

Quantum mechanically, the action given in (3.8) implies the propagator

$$\langle b_{zz}(z)\, c^w(w)\rangle \ = \ \frac{1}{z-w}\ . \tag{3.15}$$

We may then treat c^z and b_{zz} as analytic tensor fields. Since the Fadde'ev-Popov procedure assigns to c^z the conformal properties of the displacement ξ^z, this field should transform as a conformal tensor of dimension (-1). The complementary antighost b_{zz} acquires dimension 2. The ghost and antighost associated with $\xi^{\bar{z}}$, $c^{\bar{z}}$ and $b_{\bar{z}\bar{z}}$, transform as anti-analytic tensors with dimensions (-1) and 2. From here on, I will drop the tensor indices and refer to these fields as $c(z)$, $b(z)$, $\bar{c}(\bar{z})$, and $\bar{b}(\bar{z})$.

The energy-momentum tensor for c and b can be reconstructed from the requirement that it reproduce the operator product (2.27) with each of c and b, assigning these field the correct dimensions. The result is

$$T(z) \;=\; -2b\partial_z c \;-\; \partial_z bc \;. \tag{3.16}$$

One may check that the single and double pole terms in the operator product of this $T(z)$ with itself are given in accordance with (2.33). More generally, we might imagine a system with anticommuting analytic tensor fields

$$\hat{b}\;, \quad \text{dimension } (j)\;; \qquad \hat{c}\;, \quad \text{dimension } (1-j)\;, \tag{3.17}$$

always with the propagator (3.15). This system is described by

$$T(z) \;=\; -j\hat{b}\partial_z\hat{c} \;+\; (1-j)\partial_z\hat{b}\hat{c}\;. \tag{3.18}$$

It is not difficult to work out the central charge of the $\hat{b}$, $\hat{c}$ system for a general value of j. We must simply compute the c-number term in the operator product of (3.18) with itself:

$$\begin{aligned} T(z)\,T(w) \;&= j^2\left\langle \hat{b}(z)\partial_w\hat{c}\right\rangle\left\langle \partial_z\hat{c}\hat{b}(w)\right\rangle \;+\; (j-1)^2\left\langle \partial_z\hat{b}\hat{c}(w)\right\rangle\left\langle \hat{c}(z)\partial_w\hat{b}\right\rangle \\ &+ j(j-1)\Big\{\left\langle \hat{b}(z)\hat{c}(w)\right\rangle\left\langle \partial_z\hat{c}(z)\partial_w\hat{b}\right\rangle \;+\; \left\langle \partial_z\hat{b}(z)\partial_w\hat{c}\right\rangle\left\langle \hat{c}(z)\hat{b}(w)\right\rangle\Big\} \\ &= j^2\left(\frac{-1}{(z-w)^4}\right) \;+\; (j-1)^2\left(\frac{-1}{(z-w)^4}\right) \\ &\quad +\; 2j(j-1)\left[\frac{-2}{(z-w)^2}\right]\left[\frac{1}{z-w}\right] \\ &= -\frac{(6j^2-6j+1)}{(z-w)^4}\;. \end{aligned} \tag{3.19}$$

Thus, we find for the central charge of anticommuting tensor fields:

$$c \;=\; -2C_j \;\equiv\; -2\cdot(6j(j-1)\;+\;1)\;. \tag{3.20}$$

If $\hat{b}$ and $\hat{c}$ had been commuting fields, the propagator (3.15) would have implied

$$\left\langle \hat{c}(z)\,\hat{b}(w)\right\rangle \;=\; -\frac{1}{z-w}\;. \tag{3.21}$$

and the change of sign in this equation would have induced a change of sign in

the final answer. Thus, for commuting tensor fields

$$c = +2C_j \,, \tag{3.22}$$

where C_j is just as given in (3.20).

Setting $j = 2$ in (3.20), we find that the system of reparametrization ghosts has $c = -26$. This cancels the contributions from $X^\mu(z)$ only if the string is embedded in a space of 26 dimensions. In that case, however, the conformal field theory on the string is conformally invariant in the strong sense described at the beginning of this section: The complete functional integral is unchanged, even by overall multiplicative factors, under a conformal transformation.

3.2. Spinors and Ghosts of the Superstring

From the formalism we have developed, it is not difficult to give a parallel discussion of the embedding dimension of the superstring. The superstring is obtained from the ordinary string by extending the geometrical invariance on the world sheet from local reparametrization invariance to local supersymmetry and replacing each coordinate field X^μ by a supermultiplet (X^μ, Ψ^μ), where Ψ^μ is a Majorana fermion. Let us first discuss this new matter field, then turn to the new ghosts. A Majorana fermion in 2 dimensions has the action

$$\int \mathcal{L} = \frac{1}{4\pi} \int d^2z \, \overline{\Psi} \gamma \cdot \partial \Psi \,, \tag{3.23}$$

where, because of the Majorana nature of Ψ, $\overline{\Psi} = \Psi^T \gamma^0$. If we introduce the explicit representation of Euclidean Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \Psi = \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} , \tag{3.24}$$

this takes the form

$$\mathcal{L} = \frac{1}{4\pi} \int d^2z \, \left[\psi \partial_{\overline{z}} \psi + \overline{\psi} \partial_z \overline{\psi} \right] . \tag{3.25}$$

The dynamics of the field ψ is almost exactly that of a $\hat{b}$, $\hat{c}$ system. We can make the connection explicit by combining two species of Majorana fermions to form

$$\hat{b} = \psi^1 + i\psi^2 \,, \qquad \hat{c} = \psi^1 - i\psi^2 \,. \tag{3.26}$$

The new fields are just the positive-chirality part of a Dirac fermion and its complex conjugate. (The negative-chirality part of the Dirac field is anti-analytic

and may be built from $\overline{\psi}^1, \overline{\psi}^2$.) These fields have an action proportional to $\int \hat{b}\partial_{\overline{z}}\hat{c}$ and thus a propagator of the form of (3.15). Since a fermion field in 2 dimensions has dimension $\frac{1}{2}$, two Majorana fermions form a $\hat{b}$, $\hat{c}$ system with $j = \frac{1}{2}$ (plus the corresponding anti-analytic system). This system of two Majorana fermions has $c = 1$. We might have guessed this result from the well-known property that a Dirac fermion in 2 dimensions can be bosonized, that is, replaced by a equivalent system containing one real scalar field. Bosonization will come to play a major role in our analysis, beginning in the next section.

In addition to the new matter fields, we find that the superstring has new ghosts corresponding to the gauge-fixing of the new geometrical invariances of this theory. Without unravelling the whole structure of these new transformations, we can easily guess that they will correspond, in conformal coordinates, to the transformation of the world-sheet gravitino field by a spinor parameter:

$$\delta\lambda_{\overline{z}\overline{\zeta}} = \partial_{\overline{z}}\eta^{\zeta} , \tag{3.27}$$

and the corresponding transformation with an analytic derivative. In (3.27), the (lowered) indices ζ, $\overline{\zeta}$ denote 2-dimensional spinor indices corresponding to the upper and lower components of (3.24). Since η is an anticommuting parameter, it must be replaced by a commuting ghost field γ^{ζ}; this field transforms as a conformal tensor of dimension $(-\frac{1}{2})$. This field will have the action

$$\int \mathcal{L} = \frac{1}{2\pi}\int d^2z\, \beta_{z\zeta}\partial_{\overline{z}}\gamma^{\zeta} , \tag{3.28}$$

where I have introduced an antighost $\beta_{z\zeta}$, also a commuting field, of dimension $\frac{3}{2}$. These two fields, which I will refer to henceforth as $\gamma(z)$ and $\beta(z)$, form a system of commuting fields with $j = \frac{3}{2}$. Thus, they contribute

$$c = +2C_{j=\frac{3}{2}} = 11 . \tag{3.29}$$

These fields have anti-analytic counterparts $\overline{\gamma}(\overline{z})$, $\overline{\beta}(\overline{z})$ which contribute to the central charge of the Virasoro operators $\overline{L}_n$.

Adding the contributions to c from all of the fields on the world sheet of the superstring—coordinates X^{μ}, fermions ψ^{μ} (taken in pairs), anticommuting ghosts b, c, and commuting ghosts β, γ—we find

$$c = D + \frac{1}{2}D - 26 + 11 = \frac{3}{2}(D - 10) . \tag{3.30}$$

The superstring is then conformally invariant in the strong sense required for string theory when it is embedded in a space of 10 dimensions.

The restriction of string theories to a particular space-time dimension is a striking requirement, unusual in the formulation of a physical theory. Much effort has been spent trying to answer the question of whether string theories are well-defined outside of this critical dimension, with results which are so far inconclusive. For the remainder of this set of lectures, I will restrict myself to working in the critical dimension. We will see that the peculiar choices of $d = 26$ and $d = 10$ actually work miraculous simplifications in many aspects of the physics of strings.

4. Vertex Operators and Tree Amplitudes

Now that we have set up the dynamics of fields on the string world surface in terms of conformal field theory, we are ready to construct the spectrum of states of the string and the interactions between these states predicted by the theory. We will find that both of these features of string dynamics may be presented in a most natural way in the language of conformal field theory.

4.1. THE BRST CHARGE

Before beginning this discussion, however, we need one more piece of technical apparatus. In the previous section, we introduced into the string world sheet the ghost fields $b(z)$, $c(z)$, noting as we did this that these fields belong to a quantum theory with negative metric. The excitations of the ghost field will naturally become a part of the spectrum of states of the string. It is important, though, that we should not find negative metric states propagating on the world sheet as a result of a physical scattering process. Actually, negative metric states can arise not only from the ghost excitations but also also from the longitudinal and timelike excitations of $X^\mu(z)$. These modes of excitation were explicitly eliminated in the light-cone gauge treatment of the string dynamics presented in Michael Green's lectures, but they are still present in the covariant formalism that I have been constructing. In order to disentangle these dangerous states, we will need some mathematical tool which will enable us to distinguish physical from unphysical modes of excitation.

The standard tool for identifying and controlling the unphysical states of a covariantly quantized field theory is the ghost charge of Becchi, Rouet, Stora,[16] and Tyupin[17] (BRST). This is a nilpotent charge—

$$Q^2 = 0 \tag{4.1}$$

—which is Hermitian, commutes with the Hamiltonian, raises the ghost number by one unit, and annihilates all ghost-free, gauge invariant states. Such a

charge can be constructed in any covariant field theory quantized in a way which introduces ghosts.*

Once the BRST charge Q has been constructed, it can be applied in the following way: The relation (4.1) implies that every eigenstate of the Hamiltonian is acted on by Q as a member of one of the following types of multiplets:

$$\begin{aligned} \text{singlet}: &\quad Q\,|\psi_0\rangle = 0\,; \\ \text{doublet}: &\quad Q\,|\psi_1\rangle = |\psi_2\rangle \qquad Q\,|\psi_2\rangle = 0\,. \end{aligned} \tag{4.2}$$

Since Q is Hermitian, $|\psi_2\rangle$ has zero norm: $\langle\psi_2|\psi_2\rangle = \langle\psi_1|Q^2|\psi_1\rangle = 0$. The gauge-invariant, ghost-free states should be BRST singlets; it is necessary to check in detail for each theory that the only 1-particle states which are BRST singlets are of this form, and that all such states have positive norm. Let us assume that this is true. Then the initial state in any scattering process will satisfy $Q\,|\psi_{in}\rangle = 0$. But then, since $[Q, H] = 0$, any state obtained by time-evolving this state will also be annihilated by Q. Thus, the final state of a scattering process must be linear combination of BRST singlets and zero-norm states. If we take a matrix element of this state to compute the S-matrix, the zero-norm pieces disappear, and we find

$$\langle\psi_f|\,S\,|\psi_i\rangle = \langle\psi_{f,0}|\,S\,|\psi_{i,0}\rangle\ , \tag{4.3}$$

where the states on the right-hand side are projected onto BRST singlets. Thus, the properties of Q allow one to prove that, if S is unitary on the full Hilbert space, as is guarranteed by Hamiltonian evolution, it is also unitary when restricted to BRST singlets. No probability disappears into ghostly states which transform nontrivially under Q.

Now that I have discussed the use of Q, I would like to describe roughly how to define Q in any given theory. This discussion abstracts a general construction due to Fradkin and Vilkoviskii.[19] Consider a theory with gauge invariances generated by charges G_i. Let b_i, c^j represent ghost and antighost fields satisfying $\{b_i, c^j\} = \delta_i{}^j$. Then we can begin the construction of Q by writing $Q = c^iG_i + \ldots$; This expression annihilates gauge-invariant states and otherwise has the form of a gauge transformation with gauge parameter c^i. Now we need only complete this expression to form a nilpotent charge by adding a piece which will act nontrivially

* A beautiful and detailed review of the BRST charge and its uses in field theory has been given by Kugo and Ojima.[18]

on states with ghosts. A suitable choice is given by writing

$$Q \;=\; c^i G_i \;+\; \frac{1}{2} f_{ij}{}^k c^j c^i b_k \;, \tag{4.4}$$

where the $f_{ij}{}^k$ are the structure constants of the algebra of the charges G_i. We may compute

$$\begin{aligned} Q^2 \;&=\; c^i c^j G_i G_j \;+\; \frac{1}{2} f_{ij}{}^k c^j c^i \{b_k, c^\ell\} G_\ell \;+\; \frac{1}{8} f_{ij}{}^k f_{\ell m}{}^n \{c^j c^i b_k, c^m c^\ell b_n\} \\ &=\; \frac{1}{2} c^i c^j f_{ij}{}^k G_k \;+\; \frac{1}{2} f_{ij}{}^k c^j c^i G_k \;+\; \frac{1}{4} f_{ij}{}^k f_{\ell k}{}^m c^j c^i c^\ell b_n \\ &=\; \qquad 0 \qquad + \qquad \frac{1}{4} f_{[ij}{}^k f_{\ell]k}{}^n (c^j c^i c^\ell b_n) \;. \end{aligned} \tag{4.5}$$

The remaining term vanishes by the Jacobi identity.

To make this construction more concrete, one must, in any specific system, assign appropriate values to the charges and ghost operators given above. In string theory, we clearly would like to identify the G_i with the conformal generators L_n and the ghost operators with the Fourier components of $b(z)$ and $c(z)$. This program is complicated by two features, the fact that the gauge algebra, first, is infinite-dimensional, and, second, may contain a central charge. However, let us overlook these issues for the moment and try to construct a Q of the form $Q = \sum c_n L_n + \ldots$. A suitable completion is given by interpreting the structure constants which appear in (4.4) as arising from the action of T_{zz} on the the ghosts themselves. Thus, for the bosonic string, we write

$$Q \;=\; \oint \frac{dz}{2\pi i} \;:\; c(z)[T^{(X)}(z) \;+\; \frac{1}{2} T^{(b,c)}(z)] \;:\;, \tag{4.6}$$

where $T^{(b,c)}$ is given by (3.16). The anti-analytic fields give a second BRST charge

$$\overline{Q} \;=\; \oint \frac{d\bar{z}}{2\pi i} \;:\; \bar{c}(\bar{z})[\overline{T}^{(X)}(\bar{z}) \;+\; \frac{1}{2} \overline{T}^{(\bar{b},\bar{c})}(\bar{z})] \;: \tag{4.7}$$

built using the anti-analytic components of the ghost fields.

To check the nilpotency of this Q, we represent $Q^2 \;=\; \frac{1}{2}\{Q, Q\}$ as the difference of correlation functions in which one contour lies just outside, then just inside, the other. This reduces the computation to the analysis of singularities

in the operator product of the two integrands. Using the explicit form of $T^{(b,c)}$ and the general expansion (2.34), it is not hard to show that the anticommutator vanishes if the combined system of X, b, c has total $c = 0$. (An analogous argument implies that $\overline{Q}^2 = 0$.) Here again, the choice of the critical dimensionality plays an important role in simplifying the formalism.

4.2. Bosonization of the Ghosts

Certain aspects of the BRST formalism for the string are made clearer if one introduces an alternative representation of the b, c system in terms of boson operators. This bosonization of fermions is a familiar feature of the physics of 2-dimensional systems.[20,21] It will soon be clear that the boson-fermion correspondences are clarified, and new generalizations are suggested, by the notation of conformal field theory. Let me now review the conventional bosonization in this language, and then generalize to the bosonization of the reparametrization ghosts.

In the previous section, I noted that a massless complex fermion (or two Majorana fermions) in 2 dimensions gave the same central charge $c = 1$ as a real boson field. I will now argue that these theories also give identical results for correlation functions, if we make the correspondence

$$\psi^1 + i\psi^2 \;\leftrightarrow\; \sqrt{2}e^{i\phi(z)} , \qquad \psi^1 - i\psi^2 \;\leftrightarrow\; \sqrt{2}e^{-i\phi(z)} . \tag{4.8}$$

If ϕ is taken to be an ordinary free boson field, (2.29) implies that both of the exponentials of this field written in (4.8) have dimension $\frac{1}{2}$. This observation and the fact that the ψ system has $c = 1$ imply that, in correlation functions of fermion operators with factors of $T(z)$, the singularities as operators $T(z)$ approach fermion operators or one another are matched if we replace operators built from ψ^1, ψ^2 with the corresponding operator built from ϕ. Using (2.30), we can compute the singularity as two of the exponentials of ϕ approach one another:

$$\sqrt{2}e^{i\phi(z)}\,\sqrt{2}e^{-i\phi(w)} \;\sim\; 2\exp\bigl(-\log(z-w)\bigr) \;=\; \frac{2}{(z-w)} ; \tag{4.9}$$

this properly reproduces the singularity of the corresponding product of fermions. Thus, these correlation functions are analytic functions with the same singularities and so must be identical. The system of two fermions is therefore physically equivalent to the bosonic theory built from ϕ. This is the free-field limit of the conventional bosonization of fermions in 2 dimensions.

Describing the reparametrization ghost system by bosons is obviously more of a challenge, since this system has $c = -26$. I will show, however, that this system is equivalent to a theory containing one real boson field σ. If we follow the argument just given, we must find elements of the σ theory which allow us to match the central charge, the dimensions of b and c, and the singularity of the b–c propagator. Begin with the central charge. Let us choose as a trial form for $T(z)$ the somewhat more general dimension 2 operator

$$T(z) \;=\; -\frac{1}{2}(\partial_z\sigma)^2 \;+\; A\partial_z^2\sigma \;, \tag{4.10}$$

where A is a constant to be determined. This gives

$$\begin{aligned} T(w)\,T(z) \;&\sim (-\tfrac{1}{2})^2\big(\langle\partial_w\sigma\,\partial_z\sigma\rangle\big)\cdot 2 \;+\; A^2\,\langle\partial_w^2\sigma\,\partial_z^2\sigma\rangle \\ &\qquad +\ (\text{less singular}) \\ &\sim\; \frac{1}{2}\,\frac{1}{(w-z)^4} \;+\; A^2\,\frac{2\cdot 3}{(w-z)^4} \;+\; \cdots \end{aligned} \tag{4.11}$$

Thus, we require $12A^2 + 1 = -26$, or $A^2 = -9/4$. We can rectify the sign in this relation by changing the sign of the kinetic energy term in the σ action and in $T(z)$. This gives σ the wrong metric, but that is only to be expected of a field which describes a ghost. The σ propagator then takes the form

$$\langle\sigma(z)\sigma(w)\rangle \;=\; +\log(z-w)\;. \tag{4.12}$$

With this choice,

$$T^{(\sigma)}(z) \;=\; +\frac{1}{2}(\partial_z\sigma)^2 \;+\; \frac{3}{2}\,\partial_z^2\sigma \tag{4.13}$$

satisfies (2.34) with $c = -26$.

By analogy to (4.8), we might seek correspondents for b and c which are exponentials of σ. Let us first compute

$$\begin{aligned} T(w)\;e^{\alpha\sigma(z)} \;&\sim\; \left[\frac{1}{2}\big(\langle\partial_w\sigma(w)\;\alpha\sigma(z)\rangle\big)^2 \;+\; \frac{3}{2}\,\langle\partial_w^2\sigma(w)\;\alpha\sigma(z)\rangle\right] e^{\alpha\sigma(z)} \;+\; \cdots \\ &\sim\; \left[\frac{1}{2}\alpha^2 - \frac{3}{2}\alpha\right]\;\frac{1}{(w-z)^2}\;e^{\alpha\sigma(z)} \;+\; \cdots \end{aligned} \tag{4.14}$$

This implies that $e^{\alpha\sigma(z)}$ is a conformal tensor of dimension $\frac{1}{2}\alpha(\alpha-3)$. Thus, we

are motivated to identify

$$b(z) \leftrightarrow e^{-\sigma(z)}, \qquad c(z) \leftrightarrow e^{\sigma(z)}, \tag{4.15}$$

since the two exponentials have dimension 2, (-1), respectively. These operators also reproduce the b–c operator product

$$e^{-\sigma(z)}\, e^{\sigma(w)} \sim \exp\bigl(-\log(z-w)\bigr) = \frac{1}{z-w}. \tag{4.16}$$

Thus, we may reproduce any correlation function of b, c, and $T^{(b,c)}$ by operators built from σ.

I would now like to rewrite the BRST charge in this bosonized language. This requires some care in the definition of operator products. Thoughout my discussion, I have been assuming that all products of operators at the same point are normal-ordered. The normal-ordering of b and c operators, however, is not the same as the normal-ordering of the corresponding σ operators, and so we must be careful when we convert complex operators from one picture to the other. Let me, then, give the explicit definition of the 3-ghost operator which appears in (4.6):

$$\begin{aligned} :c(z)\,T^{(b,c)}(z): = \lim_{z\to w} \Bigl[\{ :c(z): :T^{(b,c)}(w): \} \\ + \bigl(\frac{1}{(z-w)^2} c(w) + \frac{2}{(z-w)} \partial_w c\bigr)\Bigr]. \end{aligned} \tag{4.17}$$

The last two terms cancel the singular terms resulting from contraction of $c(z)$ with the factors of $b(w)$ in $T^{(b,c)}$; only then is the indicated limit smooth. If we bosonize each operator on the right-hand side of (4.17), we obtain

$$
\begin{aligned}
:c(z)\,T^{b,c}(z): \;&=\; \lim_{z\to w}\;\Big[e^{\sigma(z)}\cdot\big(\frac{1}{2}(\partial_w\sigma)^2 \;+\; \frac{3}{2}\partial_w^2\sigma\big) \\
&\qquad\qquad + \big(\frac{1}{(z-w)^2}e^{\sigma(w)} \;+\; \frac{2}{(z-w)}\partial_w e^{\sigma(w)}\big)\Big] \\
&=\; \lim_{z\to w}\;\Big[:e^{\sigma(z)}\Big\{-\frac{1}{(z-w)^2} \;-\; \frac{1}{(z-w)}\,\partial_w\sigma \\
&\qquad\qquad + \;\frac{1}{2}(\partial_w\sigma)^2 \;+\; \frac{3}{2}\partial_w^2\sigma\Big\}: \\
&\qquad\qquad + \big(\frac{1}{(z-w)^2}e^{\sigma(w)} \;+\; \frac{2}{(z-w)}\partial_w e^{\sigma(w)}\big)\Big] \\
&=\; \lim_{z\to w}\;\Big[:\Big\{-\frac{1}{(z-w)^2} \;-\; \frac{2}{(z-w)}\,\partial_w\sigma \\
&\qquad\qquad - \;\frac{3}{2}(\partial_w\sigma)^2 \;-\; \frac{1}{2}\partial_w^2\sigma \;+\; \frac{1}{2}(\partial_w\sigma)^2 \;+\; \frac{3}{2}\partial_w^2\sigma\Big\}\,e^{\sigma(w)}: \\
&\qquad\qquad + \big(\frac{1}{(z-w)^2}e^{\sigma(w)} \;+\; \frac{2}{(z-w)}\partial_w e^{\sigma(w)}\big)\Big] \\
&=\; e^{\sigma(w)}\{-(\partial_w\sigma)^2+\partial_w^2\sigma\} \\
&=\; e^{\sigma(w)}\left\{(\partial_w\sigma)^2 \;+\; \partial_w^2\sigma\right\} \;-\; \partial_w\sigma\,\partial_w\big(e^{\sigma}\big)\;.
\end{aligned}
\tag{4.18}
$$

Since Q involves the integral of this expression, we may integrate the last term by parts; then the integral of (4.18) falls into the form $2\cdot\oint dw\;:e^{\sigma(w)}T^{(\sigma)}(w):$. The factor 2 is just what we need to convert (4.6) to

$$
Q \;=\; \oint\frac{dw}{2\pi i}\;:\,e^{\sigma(w)}\;T^{(tot)}(w)\;:\,, \quad \text{with } T^{tot}=T^{(X)}+T^{(\sigma)}\;. \tag{4.19}
$$

From this expression it is even more straightforward to apply the trick described below (4.6) and show that $Q^2=0$ if $T^{(tot)}$ has zero central charge.

4.3. STRING SCATTERING AMPLITUDES

We are now ready to construct the amplitude for scattering of strings. To do this, we will pursue the following strategy: We will first identify asymptotic single-particle states of string which can represent the initial and final states of a scattering process. To avoid the propagation of unphysical modes, we will insist that these states are BRST invariant. Then we will cast these states into the form of excitations on the world sheet, and allow those excitations to propagate and eventually overlap with one another. The overlap of the time-evolved initial and final states defines the scattering amplitude.

To define asymptotic states, we apply in a powerful way the conformal invariance of the theory. Using the conformal mapping

$$z - z_0 = e^w , \tag{4.20}$$

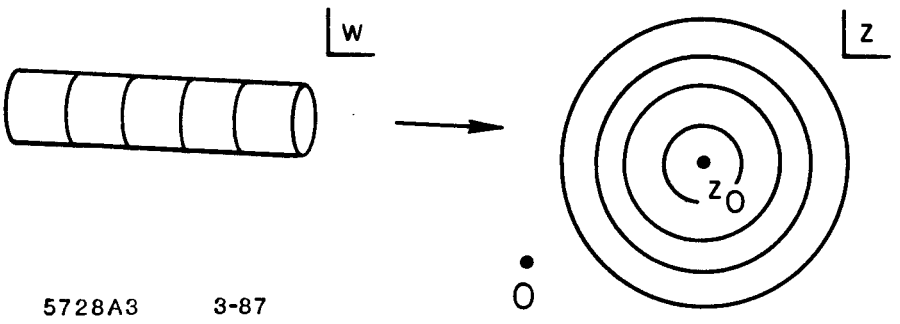

Figure 3. Identification of an asymptotic region of the closed string with the neighborhood of a point z_0 on the conformal plane.

according to the construction (2.3), we can map the asymptotic region of an infinitely extended closed-string world surface into the neighborhood of the point $z = z_0$. This transformation is shown in Fig. 3. In the discussion to follow, I will assume for simplicity that $z_0 = 0$. This entails no loss of generality; the Hamiltonian evolution described in Section 2, defined by dilatations outward from $z = 0$, could equally well have been set up about any other point.

If we put no operator at the end of the cylinder shown in Fig. 3, what propagates in from infinity is the ground state of the string in the sector with vacuum quantum numbers. Mapping this situation conformally to the z plane, the observation reads as follows: If we put *no* operator at $z = z_0$, this defines a particular state with vacuum quantum numbers as the initial state in the time evolution of radial quantization. Let us refer to this state as $|0\rangle$.

To define any other asymptotic state, we would apply some operator asymptotically on the string, or, equivalently, at $z = z_0$. The set of operators which might be used in this construction is most simply discussed in terms of the Fourier components of conformal fields, evaluated on a circle about z_0. If we translate to $z_0 = 0$, the definition of the Fourier components is that given in (2.42). Let us look more closely at the first part of this definition. Using this equation and the correspondence between operators and correlation functions, we may write

$$t_n |0\rangle = \left\langle \cdots \oint \frac{dz}{2\pi i} z^{n+d_t-1} t(z) \right\rangle , \tag{4.21}$$

where the contour is a small circle about $z = 0$. The operator ordering on the left-hand side gives the radial time ordering for the right-hand side; thus, if t_n is applied directly to the vacuum on the left, there should be no other operator inside the contour on the right. But then, if $n + d_t - 1 \geq 0$, we can contract this contour to $z = 0$ and find zero. This implies

$$t_n |0\rangle = 0 \quad \text{for} \quad n \geq 1 - d_t . \tag{4.22}$$

The moments of $t(z)$ which do not satisfy this condition may be evaluated by contour integration; one finds simply $t(0)$ and the successive derivatives of t evaluated at 0. We may view these as the creation operators for string excitations.

As an example, consider the moments of $\partial_z X^\mu$, which are conventionally denoted by

$$\alpha_n^\mu = +i \oint \frac{dz}{2\pi i} z^n \partial_z X^\mu . \tag{4.23}$$

For $n > 0$, α_n^μ is an L_0 lowering operator. All of these operators, and also α_0^μ, annihilate $|0\rangle$. It is natural to interpret the α_n^μ as lowering operators for the string normal-mode osciallations. Their counterparts α_{-n}^μ would be the corresponding raising operators. These operators may be equivalently represented as the derivatives of $\partial_z X^\mu$ evaluated at 0; for example, $\alpha_{-1}^\mu = \partial_z X(0)$. We can confirm this identification of raising and lowering operators by computing the

commutator of two α_n^μ, using the functional method which by now should be quite familiar:

$$\begin{aligned}
[\alpha_n^\mu, \alpha_m^\nu] &= (+i)^2 \left[\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right] z^n \partial_z X^\mu \, w^m \partial_w X^\nu \\
&= \oint \frac{dw}{2\pi i} \, w^m \oint \frac{dz}{2\pi i} \, z^n \, \frac{1}{(z-w)^2} \, \delta^{\mu\nu} \\
&= \oint \frac{dw}{2\pi i} \, w^m \cdot (n \, w^{n-1}) \, \delta^{\mu\nu},
\end{aligned} \tag{4.24}$$

so that

$$[\alpha_n^\mu, \alpha_m^\nu] = n \, \delta(n+m) \delta^{\mu\nu} \, . \tag{4.25}$$

The Fourier component α_0^μ is naturally interpreted the center-of-mass momentum operator of the string. With this interpretation, α_0^μ annihilates $|0\rangle$ because this state has zero momentum. To obtain a state of finite momentum, we can inject momentum into the string by applying the operator $e^{ip \cdot X}$ at z_0. We can check all of these identifications by computing

$$\begin{aligned}
\alpha_n^\mu e^{ip \cdot X(0)} |0\rangle &= +i \oint \frac{dz}{2\pi i} \, z^n \partial_z X^\mu(z) \, e^{ip \cdot X(0)} \\
&= +i \oint \frac{dz}{2\pi i} \, z^n \left\{ \left(\frac{-ip^\mu}{z} \right) e^{ip \cdot X(0)} + (\text{nonsingular}) \right\} \\
&= \begin{cases} 0 & n > 0 \\ p^\mu \, |0\rangle & n = 0 \end{cases} .
\end{aligned} \tag{4.26}$$

Indeed, the exponential shifts the center-of-mass momentum of the string while allowing all of the nonzero-frequency oscillators to remain in their ground states. We have now found that the state

$$e^{ip \cdot X(0)} \, |0\rangle \tag{4.27}$$

represents the asymptotic state of a ground-state string at momentum p^μ. Excited state of the string are represented by composite operators

$$\partial_z X^\mu e^{ip \cdot X(0)} \, , \quad \partial_z X^\mu \partial_z X^\nu e^{ip \cdot X(0)} \, , \quad \partial_z^2 X^\mu e^{ip \cdot X(0)} \, , \quad \text{etc.} \tag{4.28}$$

Operators such as those shown in (4.28) which are used to define asymptotic states of string are called *vertex operators.*

All of the excitations shown eq. (4.28) are created by analytic derivatives of X^μ; these represent left-moving excitations on a closed string. We could equally well have used anti-analytic derivatives of X^μ—$\partial_{\bar{z}} X^\mu(0)$ and higher derivatives. It follows from the generalization of the above argument to the anti-analytic sector that these operators create right-moving excitations in the asymptotic string states. In addition, it is straightforward to describe the analytic and anti-analytic ghost excitations as being created and destroyed by ladder operators, or, equivalently, by local vertex operators. By applying the manipulations of eq. (4.24), it is straightforward to show that the operator product

$$b(z)\ c(w)\ \sim\ \frac{1}{z-w} \tag{4.29}$$

implies that the Fourier components satisfy the anticommutation relations of ladder operators:

$$\{b_n,\, c_m\}\ =\ \delta(n+m)\ . \tag{4.30}$$

According to (4.22), b_n and c_m annihilate $|0\rangle$ unless $n \leq 1$, $m \leq -2$. Comparing with (2.42), we see that the b_n, c_m which do not annihilate $|0\rangle$ are precisely those which can be incorporated into vertex operators as the values of b and c and their successive derivatives at $z = 0$.

Once we have formulated a vertex operator $\mathcal{O}(0)$, we must decide whether the corresponding asymptotic state is BRST invariant. The state $|0\rangle$ is a BRST invariant state, since Q is defined as a contour integral, and this contour can be deformed to zero if it encloses no other operators. The operation $Q\,\mathcal{O}(z_0)\,|0\rangle$ is defined functionally by drawing the BRST contour about the operator $\mathcal{O}$ placed at z_0. Let us check whether this quantity vanishes.

$$Q\ \mathcal{O}(z_0)\ =\ \oint \frac{dw}{2\pi i}\, e^\sigma T(w)\, \mathcal{O}(z_0)\ . \tag{4.31}$$

To compute the operator product which gives the singularity of the integrand, assume that $\mathcal{O}(z_0)$ contains no ghost operators (as is true for the operators (4.28)) and that $\mathcal{O}(z_0)$ is a primary conformal field. Then (4.31) becomes

$$\begin{aligned} Q\ \mathcal{O}(z_0)\ &=\ \oint \frac{dw}{2\pi i}\, e^\sigma \left[\frac{d_\mathcal{O}}{(w-z_0)^2} + \frac{1}{(w-z_0)}\,\partial_z\right] \mathcal{O} \\ &=\ \left[d_\mathcal{O}(\partial_z e^\sigma)\cdot \mathcal{O}\ +\ e^\sigma \cdot \partial_z \mathcal{O}\right](z_0)\ . \end{aligned} \tag{4.32}$$

This result apparently can never vanish. However, if $d_\mathcal{O} = 1$, the final result is a

total derivative, so that

$$Q \int dz\ \mathcal{O}(z) \ = \ \int dz\ \partial_z(e^{\sigma}\mathcal{O}) \ = \ 0\ . \tag{4.33}$$

The integral of a primary conformal field of dimension 1 thus gives a representation on the complex plane of a BRST invariant asymptotic state.

How should we properly interpret this result? Up to this point, all of the intuition we have used to motivate this construction has come from the consideration of closed strings. Closed strings have a local symmetry under the action both of the L_n and the $\overline{L}_n$; thus physical states should be invariant to both Q and $\overline{Q}$. In the closed-string theory, then, asymptotic states should be associated with operator insertions of the form

$$\int dz d\bar{z}\ \mathcal{O}(z,\bar{z})\ , \tag{4.34}$$

where $\mathcal{O}$ is primary and has dimension 1 relative to $T(z)$ and also with respect to $\overline{T}(\bar{z})$. For the simplest vertex operator

$$\mathcal{O} \ = \ e^{ip\cdot X(z,\bar{z})}\ , \tag{4.35}$$

the restriction on the dimensionality implies that

$$p^2 \ = \ 2\ , \tag{4.36}$$

that is, the asymptotic state must be an on-shell state of a particle of mass $m^2 = -2$.* This state is a tachyon, and that (unfortunately) is the correct result for the ground state of the closed string.

The construction in (4.33) has, however, another interpretation. We might recognize that (4.25), by itself, is precisely the algebra of mode creation and annihilation operators for the open string. This should tempt us to interpret the analytic sector of the system we have constructed as describing the open string. Mapping back to the cylinder according to (2.3), then slicing the cylinder into two semicircular pieces, we can see that, in this interpretation, the real axis of the z plane should be identified with the boundary of the open string. The open string boundary condition that $\partial_\sigma X^\mu = 0$ at the endpoint may be

* In Michael Green's conventions, this would read: $m^2 = -8$.

given the following interpretation: Let left-moving excitations on the open string be represented by analytic functions in the upper half-plane and right-moving excitations by analytic functions in the lower half-plane. Then the open string boundary condition is simply the requirement that these functions be continuous across the real axis.

This interpretation of the analytic sector of the world sheet dynamics suggests a new interpretation of the integral of a vertex operator: The integral $\int dz\mathcal{O}(z)$ along the real axis of the z-plane represents a BRST-invariant open string asymptotic state which couples to another open string at its boundary. The simplest such state is the one associated with the vertex operator $\mathcal{O} = e^{ip\cdot X}$. The requirement that $d_{\mathcal{O}} = 1$ again implies $p^2 = 2$, so this state is again a tachyon. This is, of course, a familiar property of the ground state of the open string.

These two constructions lead to formulae for the scattering amplitudes of open and closed strings, since the transition amplitude from a set of initial to a set of final string states can be computed as the joint correlation function of the corresponding vertex operators.

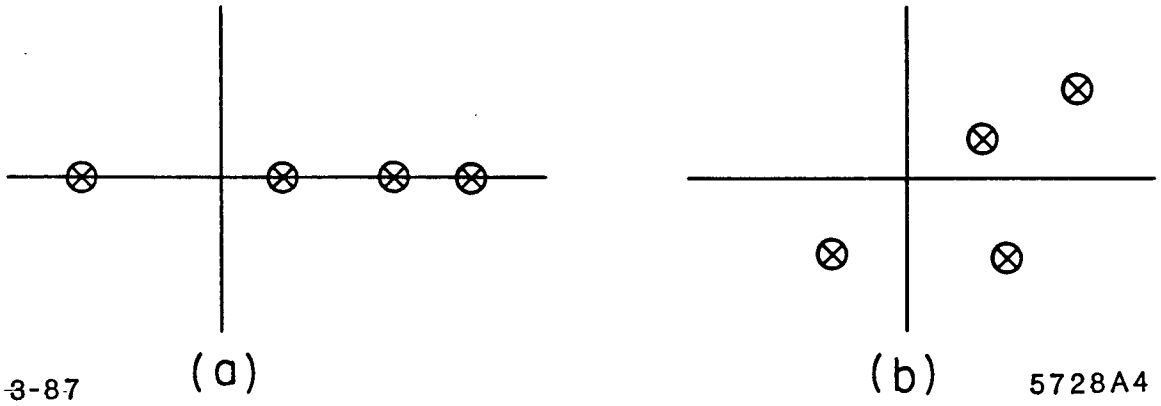

Figure 4. Correlation functions of vertex operators which give the scattering amplitudes of the (a) open and (b) closed string theories.

The open and closed string versions of this construction are illustrated in Fig. 4. We can then check our formalism by computing the scattering amplitude for four tachyons explicitly in each theory.

Begin with the scattering amplitude for open-string tachyons. This is given

by a correlation function of four vertex operators integrated along the real axis:

$$A = \left\langle \int dz_1 dz_2 dz_3 dz_4 \; e^{ip_1 \cdot X(z_1)} \; e^{ip_2 \cdot X(z_2)} \; e^{ip_3 \cdot X(z_3)} \; e^{ip_4 \cdot X(z_4)} \right\rangle . \quad (4.37)$$

As Michael Green has discussed, open strings may be given flavor quantum numbers by assigning a group theory factor (called the Chan-Paton factor) to each cyclic ordering of the operators. This factor gives the amplitude that the quantum numbers created on the string boundary by each vertex operator can be annihilated by the next operator along the line. I will ignore this factor in my discussion here, though I will assume that the cyclic order of the vertex operators is fixed. Evaluating the correlation function, we find

$$A = \int dz_1 dz_2 dz_3 dz_4 \prod_{i<j} \exp\bigl(p_i \cdot p_j \log(z_i - z_j)\bigr) . \quad (4.38)$$

The expression (4.38) is, unfortunately, not yet well defined. The problem is one that should be familiar already from Michael Green's discussion of the string scattering amplitude: The integrand of (4.38) has a group of invariances which forces the integral to diverge. A part of this divergence comes from the fact that the integrand is translation invariant. The full symmetry is the unbounded three-parameter group of fractional linear transformations with real coefficients

$$z \rightarrow \left(\frac{az+b}{cz+d} \right) , \quad (4.39)$$

exactly the group of eq. (2.39), specialized to real coefficients—$SL(2,R)$. This is in fact the group of conformal transformations which map the upper half plane onto itself 1-to-1. It is straightforward to check that the change of variables (4.39) leaves the integrand unchanged as long as each $p_i^2 = 2$. A natural way to cure this problem is to divide (4.38) by the group volume of $SL(2,R)$. Using the infinitesimal form of an $SL(2,R)$ transformation given in (2.41), we can find the Jacobian for the change of variables from any three of the (real) z_i to the (real) parameters α, β, γ:

$$\left| \frac{\partial(z_i, z_j, z_k)}{\partial(\alpha, \beta, \gamma)} \right| = (z_i - z_j)(z_i - z_k)(z_j - z_k) \quad (4.40)$$

Making this change of variables using z_1, z_2, z_4, and then cancelling the integral

over group parameters, we find the final result for A:

$$A = \int dz_3 \; (z_1 - z_2)(z_1 - z_4)(z_2 - z_4) \prod_{i<j} \exp\left(p_i \cdot p_j \log(z_i - z_j)\right) . \quad (4.41)$$

This result should be independent of the choice of z_1, z_2, z_4, as long as the four points are in the correct cyclic order. If we assign $z_1 \to \infty$, $z_2 = 1$, $z_4 = 0$, always using $p_i^2 = 2$ and momentum conservation, (4.41) simplifies to the form

$$A = \int_0^1 dz \; (1-z)^{p_2 \cdot p_3} \, z^{p_3 \cdot p_4} . \quad (4.42)$$

This is the famous amplitude of Veneziano[22] which was in fact the first result in string theory.

The scattering amplitude of the closed string theory may be computed by following an analogous set of steps. One begins from the expression

$$A = \left\langle \int d^2z_1 d^2z_2 d^2z_3 d^2z_4 \; e^{ip_1 \cdot X(z_1)} \; e^{ip_2 \cdot X(z_2)} \; e^{ip_3 \cdot X(z_3)} \; e^{ip_4 \cdot X(z_4)} \right\rangle , \quad (4.43)$$

where the integrals now run over the whole complex plane. The correlation function may be evaluated as in (4.38). The resulting expression is invariant to transformations of the form (4.39) with general complex coefficients; this is the full group $SL(2,C)$. This group is in fact the group of all 1-to-1 conformal mappings of the whole z plane onto itself. The amplitude (4.43) can be made well-defined by dividing by the volume of this group. The Jacobian needed in this construction is the absolute square of (4.40). The result is

$$A = \int d^2z_3 \; |(z_1 - z_2)(z_1 - z_4)(z_2 - z_4)|^2 \prod_{i<j} \exp\left(2p_i \cdot p_j \log|z_i - z_j|\right) . \quad (4.44)$$

Assigning z_1, z_2, z_4 to ∞, 1, 0, we find

$$A = \int d^2z \; |1-z|^{2p_2 \cdot p_3} \, |z|^{2p_3 \cdot p_4} , \quad (4.45)$$

the Virasoro-Shapiro[23,24] closed string amplitude.

To conclude this discussion of string scattering amplitudes, I would like to discuss briefly the problem of constructing BRST invariant vertex operators for higher states of the string spectrum. The open-string vector states $\zeta^\mu \alpha^\mu_{-1} |0\rangle$ are created by vertex operators

$$\mathcal{O}_\zeta(z) \;=\; \zeta^\mu \partial_z X^\mu(z) e^{ip\cdot X(z)} \;. \tag{4.46}$$

We must check not only that this operator has the correct dimension but also that it is a primary conformal field. To do this, compute its operator product expansion with $T(z)$:

$$\begin{aligned} T(w)\;\mathcal{O}_\zeta(z) \;\sim\; & -\frac{1}{2}\langle \partial_w X^\mu \; ip\cdot X(z)\rangle \langle \partial_w X^\mu \; \zeta^\nu \partial_z X^\nu\rangle \; e^{ip\cdot X(z)} \cdot 2 \\ & - \frac{1}{2}\Big(\langle \partial_w X^\mu \; ip\cdot X(z)\rangle\Big)^2 \zeta^\nu \partial_z X^\nu \, e^{ip\cdot X(z)} \\ & - \frac{1}{2}\partial_w X^\mu \langle \partial_w X^\mu \; \zeta^\nu \partial_z X^\nu\rangle \; e^{ip\cdot X(z)} \cdot 2 \;+\; \ldots \\ \sim\; & \frac{ip\cdot\zeta}{(w-z)^3}\, e^{ip\cdot X(z)} \;+\; \frac{p^2/2+1}{(w-z)^2}\, \zeta\cdot\partial_z X \, e^{ip\cdot X(z)} \;+\; \ldots \end{aligned} \tag{4.47}$$

The operator $\mathcal{O}_\zeta$ has dimension 1 if $p^2 = 0$, that is, if the vector particles created by this operator are massless. However, (4.47) informs us that we must impose an additional constraint in order that $\mathcal{O}_\zeta$ be primary. This is the condition $\zeta\cdot p = 0$, the familiar physical requirement of transversality. In a similar way, we may find that the tensor vertex operator

$$\eta^{\mu\nu}\, \partial_z X^\mu \partial_z X^\nu e^{ip\cdot X} \tag{4.48}$$

has dimension 1 if $p^2 = -2$, corresponding to $m^2 = 2$, but it is primary only if we satisfy as well the physical state conditions on the polarization tensor

$$p_\mu \, \eta^{\mu\nu} \;=\; 0 \;=\; \eta^\mu{}_\mu \;. \tag{4.49}$$

In general, for string modes with spin, the requirement of BRST invariance implies not only that the particle should be on its mass shell but also that the polarization should have a physically correct orientation. This is the first hint of the deep connection between BRST invariance on the world sheet and gauge invariance in space-time which we will explore in the next section.

5. Gauge Invariances of the Bosonic String

In the previous section, we formulated a set of rules for computing string scattering amplitudes at the tree level. The basic assumption of this construction was that the string scattering amplitude should involve propagation on the world sheet of physical (that is, BRST invariant) excitations. We found, however, that this condition implies that the external particles of string must also be physical propagating states on space-time. This correspondence between world-sheet and space-time properties is already remarkable in itself, but it is worth pushing the argument one step further. Our experience with local field theories with spin tells us that such theories cannot be formulated covariantly and still naturally project out unphysical polarizations unless they possess an underlying gauge invariance. We have seen also that the open string theory contains massless vector states; it would be most attractive if these were the gauge bosons of some explicit local invariance of the string theory. In this section, I will exhibit the local gauge invariances of the free string theory and clarify the relation between these invariances and the world-sheet dynamics.

5.1. More Conformal Field Theory

To carry out this analysis, we will need some additional tools from conformal field theory. In the previous section, I introduced the vacuum state $|0\rangle$, defined by putting no operator at the point $z = 0$, which forms the asymptotic past in radial quantization. To continue our discussion, it will be useful to define $\langle 0|$, and, more generally, a notion of operator adjoints and inner product appropriate to conformal field theory. All of the necessary concepts are provided in the work of BPZ.

The obvious definition of $\langle 0|$ is to put nothing at the point $z = \infty$ which forms the asymptotic future of radial quantization. Just as contour integrals which can be deformed to 0 represent operators which annihiliate $|0\rangle$, operators will annihilate this $\langle 0|$ when the corresponding contour integrals can be pushed to ∞. Conformal invariance allows us to discuss both situations at the same time, because a conformal transformation

$$z \to \tilde{z} = -\frac{1}{z} \tag{5.1}$$

interchanges past and future. BPZ proposed that this conformal transformation could in fact be taken to be the definition of the adjoint of an operator or state. This definition has the virtue that it does not interchange the analytic and anti-analytic sectors. We can check the definition by applying it to a general Fourier

component t_n:

$$\begin{aligned} t_n &= \oint \frac{dz}{2\pi i}\, z^{n+d_t-1} t(z) \\ &\longrightarrow \oint \frac{dz}{2\pi i}\, z^{n+d_t-1} \left(\frac{d\tilde{z}}{dz}\right)^{d_t} t(\tilde{z}) \\ &= \oint \frac{d\tilde{z}}{2\pi i}\, \frac{1}{\tilde{z}}\, (-\tilde{z})^{-n-d_t}\, (\tilde{z}^2)^{d_t}\, t(\tilde{z}) \\ &= (-1)^{d_t+n}\, t_{-n}\ . \end{aligned} \tag{5.2}$$

(A factor of (-1) disappears in the second step because both the z and the $\tilde{z}$ contours are taken to run counterclockwise around the unit circle.) The transformation (5.1) thus does interchange t_n and t_{-n}. We find as well the equivalence

$$t_n\,|0\rangle \;=\; 0 \qquad \leftrightarrow \qquad \langle 0|\, t_{-n} \;=\; 0\ . \tag{5.3}$$

According to (4.22), these statements apply when $n \geq 1 - d_t$.

The L_n annihilate $|0\rangle$ for $n \geq (-1)$. Reciprocally, $\langle 0|\, L_n = 0$ for $n \leq 1$. Thus, the three operators L_{-1}, L_0, L_1, and their anti-analytic counterparts $\overline{L}_{-1}$, $\overline{L}_0$, $\overline{L}_1$, annihilate both $|0\rangle$ and $\langle 0|$. This makes these operators true symmetries of conformal field theory: If U is a transformation generated by these operators—that is, if $U \in SL(2,C)$—

$$\langle 0|\phi(z_1)\,\phi(z_2)\cdots|0\rangle \;=\; \langle 0|(U\phi(z_1)U^{-1})(U\phi(z_2)U^{-1})\cdots|0\rangle \tag{5.4}$$

Thus, all conformal field theory matrix elements are invariant to $SL(2,C)$ transformations. This is the origin of the conformal invariances of the four-tachyon scattering amplitudes discussed at the end of the last section. Because of this property, the state $|0\rangle$ is often referred to as the $SL(2,C)$ invariant vacuum.

The antighost field $b(z)$ has the same dimension as $T(z)$, and therefore, in parallel to the above discussion, b_{-1}, b_0, and b_1 annihilate both $|0\rangle$ and $\langle 0|$. But then the identity following from the commutation relation (4.30) of the ghost ladder operators

$$\{b_n,\, c_{-n}\}\,|0\rangle \;=\; |0\rangle \;\neq\; 0\ ; \tag{5.5}$$

implies that the three operators c_{-1}, c_0, c_1 annihilate neither $|0\rangle$ nor $\langle 0|$. All other c_n and b_n ladder operators annihilate either $|0\rangle$ or $\langle 0|$. This set of statements

implies that the basic nonzero expectation value in the theory is

$$\langle 0|\ c_{-1}\, c_0\, c_1\ |0\rangle\ \neq\ 0\ . \tag{5.6}$$

Apparently, the adjoint operation of BPZ does not preserve ghost number; indeed, it insists that ghost number is violated by 3 units in conformal field theory calculations on the plane.

To properly understand this ghost number nonconservation, it is necessary to develop in some detail the geometrically invariant formulation of the string dynamics. Since this would take us too far afield, I will simply sketch the logic of the required argument. The ghost number current $j_z = bc$ is the fermion number current of a set of chiral fermions (of unconventional spin) in 2 dimensions. Therefore, we should expect that this current has a gravitational anomaly. Its conservation law is, in fact,

$$\partial_{\bar{z}} j_z\ =\ -\frac{3}{8\pi} R\ , \tag{5.7}$$

where my conventions correspond to $\int d^2z\sqrt{g}R = 8\pi$ for a sphere. Considered for more general 2-dimensional manifolds, this integral is a topological invariant, the Euler characteristic χ; its value is

$$\frac{1}{4\pi}\int d^2z\sqrt{g}\, R\ =\ -2(g-1)\ , \tag{5.8}$$

where g is the *genus* of the surface, the number of handles. For surfaces of the topology of a sphere, $g = 0$ and we expect (-3) units of ghost number nonconservation. Since the complex plane, including the point at infinity, has this topology, we thus obtain the nonconservation law displayed in (5.6). For surfaces with handles, ghost number $3(g-1)$ should be swallowed by the conformal field theory matrix element.

When one encounters fermion number nonconservation due to the coherent effects of anomalies, the nonconservation is normally manifested in the appearance of *zero modes* of the fermion field. These are modes of the fermion field which are localized solutions to the equations of motion. If one defines the functional integral over the fermion field by decomposing in eigenmodes of the fermion action

$$\psi(x)\ =\ \sum_n \psi_n\, \varphi_n(x)\ , \tag{5.9}$$

the zero modes $\varphi_0(x)$ are annihilated by the Dirac operator, and so the components ψ_0 which multiply these solutions do not appear in the action. The